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Generating the weakly efficient set of nonconvex multiobjective problems

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Abstract We present a method for generating the set of weakly efficient solutions of a nonconvex multiobjective optimization problem. The convergence of the method is proven and some numerical examples are encountered.

Keywords Nonconvex multiobjective problem · Weakly efficient solution · Scalarization

AMS Subject Classification 90C31

1 Introduction

Throughout this paper let us denote by $I\!R^n$ the n-dimensional Euclidean space and by $I\!R^n_+$ its positive orthant. Given $a, b \in I\!R^n$, we write $a > b$ (resp. $a \geq b$ and $a \geqq b$) when $a - b \in \mathrm{int} I\!R^n_+$ (resp. $a - b \in I\!R^n_+ \setminus \{0\}$ and $a - b \in I\!R^n_+$), where $\mathrm{int} I\!R^n_+$ stands for the interior of $I\!R^n_+$. Let A be a nonempty subset of $I\!R^n$. A point $a \in A$ is said to be an efficient point (resp., weakly efficient point) of A if there exists no $b \in A$ such that $b \geq a$ (resp., $b > a$). The sets of all efficient points and weakly efficient points of A are respectively denoted by $\mathrm{Max}(A)$ and $\mathrm{WMax}(A)$.

Let $f\colon I\!R^m \to I\!R^n$ be a vector function and $X \subseteq I\!R^m$ a nonempty set. We consider the following multiobjective optimization problem associated with f and X:

$$\begin{aligned}&\mathrm{Max}\ \ f(x)\\ &\text{subject to } x \in X.\end{aligned} \tag{VP}$$

This problem means finding a point $x_0 \in X$ such that $f(x_0)$ is an efficient point of the set $f(X)$, or in other words, there is no $x \in X$ verifying the inequality $f(x) \geq f(x_0)$. The point x_0 is called an efficient solution and the vector $f(x_0)$ is called an efficient value of (VP). The set of all efficient solutions of (VP) is denoted by $\mathrm{S}(f, X)$ and its image $\mathrm{Max}(f(X)) = f[S(f, X)]$

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is called the efficient value set of (VP). Sometimes one is interested in finding weakly efficient solutions $x_0 \in X$ in the sense that $f(x_0)$ is a weakly efficient point of the set $f(X)$. The weakly efficient solution set of (VP) is denoted by $\mathrm{WS}(f, X)$ and its image $\mathrm{WMax}(f(X))$ under f is called the weakly efficient value set of (VP). It is clear that the inclusion $\mathrm{S}(f, X) \subseteq \mathrm{WS}(f, X)$ holds and in general it is strict.

Over the last three decades various methods for solving problem (VP) have been proposed. The majority of them are aimed at obtaining one or some solutions of (VP), frequently by combining mathematical programming algorithms with the interaction of a decision maker who is responsible for choosing a suitable solution among the efficient solutions. Another class of methods attempts to approximate the entire efficient solution set $\mathrm{S}(f, X)$ or its image $\mathrm{Max}(f(X))$. The problem of finding the whole solution set of a multiobjective problem is important in applications, especially in multicriteria design and in multicriteria decision making (see [5,8,24,25]). Its solution, however, is a very difficult task. We know that identifying all optimal solutions of a scalar programming problem is numerically possible only when f and X have a special structure. In the multiobjective case, even when these data are linear, the computational demands increase so fast with problem size that most existing algorithms refuse to provide satisfactory results when the number of criteria is relatively large [2–4]. Because of the complexity of this problem, by our knowledge, relatively few works exist which fully describe numerical algorithms for finding the entire set $\mathrm{S}(f, X)$ or $\mathrm{WS}(f, X)$ apart from the classical simplex method (see [26]). For linear problems of medium size some recent methods such as Armand's lexicographic selection based simplex method [1], Benson's outcome space method [3], Kim and Luc's normal cone method [9] are quite effective in constructing all maximal faces of the set $\mathrm{S}(f, X)$ or $\mathrm{WS}(f, X)$. For nonlinear problems a number of methods have lately come to light. Detailed discussions on several existing methods can be found in the monograph by Miettinen [18] and in the survey paper by Ruzika and Wiecek [22] (see also [12–15,20,23]). Most of these methods use inner or outer approximations in order to produce as large as possible a subset of the efficient (or weakly efficient) value set. We mention here some of them which are related to the idea of the method we are going to develop. In the case of convex problems, the papers [16] and [17] offer algorithms by normal projection and duality for generating a solution set whose image converges to the set $\mathrm{WMax}(f(X))$. For nonconvex problems, the paper [6] provides a numerical algorithm to produce an evenly distributed set of points in the set $\mathrm{Max}(f(X))$. This method is easy for coding, but it is not always sure that the solution set obtained by the algorithm converges to the solution set of the multiobjective problem when the number of iterations grows to ∞. The paper [10] presents both inner and outer approximations to the weakly efficient value set (which becomes the efficient value set under a strict convexity hypothesis) of the problem, but the convergence of the method is not fully described. The paper [19] also gives a method to scalarize nonconvex multiobjective problems, but no solving algorithms are proposed.

The goal of this paper is to develop a method to generate the set $\mathrm{WS}(f, X)$ when the problem is not convex, that is either X is not convex or f is not concave, or both. Our method belongs to the class of outer approximations and is close to that of [10]. The distinction is the use of particular scalarizing functions in the approximating process which allows us to rigorously establish the convergence of the method. The choice of studying the weakly efficient set instead of the efficient set is due to technical difficulties in proving the convergence. In practical situations one is rather interested in efficient solutions than in weakly efficient solutions. The concern is, however, that the set of efficient solutions is unstable while the set of weakly efficient solutions is stable. For instance, given a convex and compact set, the limit of a convergent sequence of efficient points of the set may be not efficient, but is always weakly efficient. So, generally without specific hypothesis on the data, convergence of approximations to the efficient solution set of the problem is not valid.

The paper is structured as follows. In Sect. 2 we construct a sequence of so-called free disposal nonconvex polyhedra which converges to the free disposal hull of a given set in the positive orthant $\mathbb{R}^n_+$. In Sect. 3, we define a monotonic function associated to a nonempty subset of $\mathbb{R}^n_+$ and prove several properties of it. Particular attention is paid on the sequence of monotonic functions associated to the sequence of free disposal nonconvex polyhedra of Sect. 2. This sequence is crucial in solving Problem (VP). In Sect. 4 we propose a method to solve (VP). To do it, first we construct a sequence of nonconvex polyhedra A_k by the method of Sect. 2 which converges to the set $f(X)$, and its associated monotonic functions g_k by the method of Sect. 3. Then we solve the scalarized problems

$$\begin{aligned} &\max \quad g_k \circ f(x) \\ &\text{subject} \quad \text{to } x \in X, \end{aligned} \tag{$P(g_k)$}$$

whose optimal solutions are a part of the weakly solution set of (VP) and converges to it as k tends to ∞. The last section is devoted to some small size numerical examples to illustrate our method and show its applicability.

2 Approximation by free disposal nonconvex polyhedra

Let us denote by $\mathcal{C}$ the collection of compact sets A in $\mathbb{R}^n_+$ such that $A = \mathrm{cl}(A \cap \mathrm{int}\mathbb{R}^n_+)$. Let $P \in \mathcal{C}$. Following Debreu's terminology [7] we define the free disposal hull of P as the set $P^\lozenge := (P - \mathbb{R}^n_+) \cap \mathbb{R}^n_+$, and say that P is free disposal if it coincides with its free disposal hull. Here are some properties of the free disposal hull which are quite obvious, but useful for future analysis.

Proposition 2.1 *Let P and Q be elements of $\mathcal{C}$. Then*

(i) $P \subseteq P^\lozenge = (P^\lozenge)^\lozenge$ *and* $P^\lozenge \in \mathcal{C}$;
(ii) $P \subseteq Q^\lozenge$ *implies* $P^\lozenge \subseteq Q^\lozenge$;
(iii) $(P \cup Q)^\lozenge = P^\lozenge \cup Q^\lozenge$ *and* $(P \cap Q)^\lozenge \subseteq P^\lozenge \cap Q^\lozenge$;
(iv) $Max(P) = Max(P^\lozenge)$ *and* $WMax(P) \subseteq WMax(P^\lozenge)$;
(v) $P^\lozenge = [Max(P)]^\lozenge = [WMax(P)]^\lozenge$.

Notice that the inclusions in (iii) and (iv) may be strict, and the set $\mathrm{Max}(P)$ is nonempty because P is compact.

A free disposal set $P \subseteq \mathbb{R}^n_+$ is said to be finitely generated (or a free disposal polyhedron) if there is a finite number of vectors $a^1, \ldots, a^k \in \mathrm{int}\mathbb{R}^n_+$ such that P is exactly the free disposal hull of the set $\{a^1, \ldots, a^k\}$.

In this section we shall construct a sequence of free disposal polyhedra that approximate the free disposal hull of a given set in $\mathbb{R}^n_+$.

Let $a, b \in \mathrm{int}\mathbb{R}^n_+$ be given. Denote

$$V(b|a) = \begin{cases} \{b(i) : i = 1, \ldots, n\} & \text{if } a < b \\ \{b\} & \text{else,} \end{cases}$$

where $b(i)$ is the vector whose coordinates are those of b except for the ith one which is equal to the ith coordinate of a.

Given $\alpha \in \mathbb{R}^n_+ \setminus \{0\}$, we define

$$h_\alpha(y) := \max\{t \in \mathbb{R} : y \geqq t\alpha\}.$$

This function has been studied by several authors ([11,21]). Its application in multiobjective optimization was first given by Pascoletti and Serafini [19]. Let record some of its properties. It is clear that h_α is defined and continuous on $\mathbb{R}^n$ when α belongs to the interior of $\mathbb{R}^n_+$, and on $\mathbb{R}^n_+$ for other α. Moreover, it is continuous in both variables α and y on $(\text{int}\mathbb{R}^n_+) \times \mathbb{R}^n$ and weakly monotonic in y on its domain of definition. Recall that a function g: $A \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be monotonic (respectively, weakly monotonic) on A if $a \geq b$ implies $g(a) > g(b)$ (respectively, $a > b$ implies $g(a) > g(b)$) for every $a, b \in A$. For $y \in \mathbb{R}^n_+$, we have

$$h_\alpha(y) = \min\left\{\frac{y_i}{\alpha_i} : i \in \{1, \dots, n\}, \alpha_i \neq 0\right\}.$$

Let $A \in \mathcal{C}$. The optimal value of the problem

$$\max_{a \in A} h_\alpha(a), \tag{P_α}$$

where $\alpha \in \mathbb{R}^n_+ \setminus \{0\}$ will be denoted by t_α. It is obvious that t_α exists because A is a compact subset of $\mathbb{R}^n_+$ and $h_\alpha(\cdot)$ is continuous on $\mathbb{R}^n_+$, and that $t_\alpha > 0$ as A meets the interior of $\mathbb{R}^n_+$.

Lemma 2.2 *Let $P \subseteq \mathbb{R}^n_+$ be a free disposal set generated by $W = \{a^1, \dots, a^k\} \subseteq int\mathbb{R}^n_+$ and let $A \subseteq P$. Then the following assertions hold:*

(i) *P is generated by its efficient elements, that is $P = [Max(W)]^\diamondsuit = [Max(P)]^\diamondsuit$;*
(ii) *For each $\alpha \in Max(W)$, one has $0 < t_\alpha \leq 1$, and $t_\alpha = 1$ if and only if $\alpha \in A$;*
(iii) *For $v \in int\mathbb{R}^n_+$, the set $Q = P \cap \{y \in \mathbb{R}^n_+ : h_v(y) \leq 1\}$ is a free disposal set generated by $\cup_{i=1}^k V(a^i\,|v)$.*

Proof The first assertion is clear due to Proposition 2.1. For the second assertion, observe that for $\alpha \in$Max(W), max$\{h_\alpha(y) : y \in P\} = 1$. As $A \subseteq P$ we deduce $0 < t_\alpha \leq 1$. Moreover, if $t_\alpha = 1$, then $A \cap (\alpha + \mathbb{R}^n_+) \neq \emptyset$, which is possible only when $\alpha \in A$ because α is then an efficient point of A as well. Conversely, if $\alpha \in A$, one has $t_\alpha \geq 1$ which becomes equality because $t_\alpha \leq 1$.

For the last assertion, Proposition 2.1(iii) implies that Q is a free disposal set. Let $\alpha \in V(a^i\,|v)$ for some $i \in \{1, \dots, k\}$. Then either $\alpha = a^i$ if $v < a^i$ is not satisfied, or $\alpha = a^i(j)$ for some $j \in \{1, \dots, n\}$ when $v < a^i$. In the first case, $h_v(\alpha) \leq 1$ which implies $\alpha \in Q$. In the second case,

$$h_v(\alpha) = \min\left\{\frac{a_1^i(j)}{v_1}, \dots, \frac{a_n^i(j)}{v_n}\right\} \leq \frac{a_j^i(j)}{v_j} = 1,$$

where $a_1^i(j), \dots, a_n^i(j)$ are the coordinates of $a^i(j)$. This and the fact that $\alpha \in P$ show that $\alpha \in Q$. Conversely, let $y \in Q$. There is $i \in \{1, \dots, k\}$ such that $y \in \{a^i\}^\diamondsuit$ and $h_v(y) \leq 1$. If v does not satisfy $v < a^i$, then $y \in (V(a^i|v))^\diamondsuit = \{a^i\}^\diamondsuit$ by definition. If $v < a^i$, then $v_j < a^i_j$ for $j = 1, \dots, n$. On the other hand, let $i_0 \in \{1, \dots, n\}$ be such that

$$\frac{y_{i_0}}{v_{i_0}} = \min\left\{\frac{y_i}{v_i} : i = 1, \dots, n\right\} \leq 1.$$

Then $y_{i_0} \leq v_{i_0}$ and we derive $y \leq a^i(i_0)$, that is $y \in a^i(i_0)^\diamondsuit$. This completes the proof. □

We now construct by induction a sequence of finitely generated free disposal sets A_k which are outer approximations of the free disposal hull of a given set $A \in \mathcal{C}$. For the initialization step ($k = 1$), we solve the following scalar problem

$$\max_{a = (a_1, \dots, a_n) \in A} a_i, \tag{P_0}$$

for $i = 1, \ldots, n$. Let α_i^0 be the optimal values which exist because A is compact, and let $\alpha^0 = (\alpha_1^0, \ldots, \alpha_n^0)$. This point is the supremum of A and is also known in some literature as ideal point. Note that $\alpha^0 \in \mathrm{int}\mathbb{R}^n_+$. Define

$$A_1 := [0, \alpha_1^0] \times \cdots \times [0, \alpha_n^0]$$
$$W_1 := \{\alpha^0\}.$$

It is clear that A_1 is a free disposal set generated by W_1=Max(A_1) (Lemma 2.2(i)). For $\alpha \in W_1$, solve the problem (P_α) and find the optimal value t_α. Define

$$V_1 := W_1 \setminus A^\diamondsuit = W_1 \setminus \{\alpha \in W_1 : t_\alpha = 1\},$$

in which the second equality follows from Lemma 2.2. Assume that A_k, W_k and V_k have already been constructed. If $V_k = \emptyset$, set $A_{k+1} = A_k$. If $V_k \neq \emptyset$, set

$$A_{k+1} = A_k \cap \left\{y \in \mathbb{R}^n_+ : h_\alpha(y) \leq t_\alpha, \alpha \in V_k\right\}.$$

Notice that $\alpha \in \mathrm{int}\mathbb{R}^n_+$ by induction and $t_\alpha > 0$ because $A \cap \mathrm{int}\mathbb{R}^n_+ \neq \emptyset$. Then the sets W_{k+1} and V_{k+1} are given by

$$W_{k+1} = \mathrm{Max}(A_{k+1}) \quad \text{and} \quad V_{k+1} = W_{k+1} \setminus A^\diamondsuit.$$

According to Lemma 2.2 it is clear from the construction that A_k is a free disposal set generated by W_k. To give some more properties on the sets A_k let us recall the concept of convergence with respect to the Hausdorff distance of closed sets. Let A_1 and A_2 be two closed sets in $\mathbb{R}^n$. The Hausdorff distance between them is defined by

$$d(A_1, A_2) = \inf\{t > 0 : A_1 \subseteq A_2 + tB_n, A_2 \subseteq A_1 + tB_n\},$$

where B_n is the closed unit ball of $\mathbb{R}^n$. Let $\{D_k\}_{k=1}^\infty \subseteq \mathbb{R}^n$ be a sequence of nonempty closed sets. We say that it H-converges to a closed set D and write $\lim_{k\to\infty} D_k = D$ if $\lim_{k\to\infty} d(D_k, D) = 0$.

Theorem 2.3 *The following assertions hold:*

(i) $A^\diamondsuit \subseteq A_{k+1} \subseteq A_k$;
(ii) $A_{k+1} = A_k \cap \left\{y \in \mathbb{R}^n_+ : h_\alpha(y) \leq t_\alpha, \alpha \in W_k\right\}$;
(iii) $V_k = W_k \setminus \{\alpha \in W_k : t_\alpha = 1\}$;
(iv) *If for some k it is $V_k = \emptyset$, then $A^\diamondsuit = A_k$;*
(v) $(\lim_{k\to\infty} A_k) \cap int\mathbb{R}^n_+ = A^\diamondsuit \cap int\mathbb{R}^n_+$.

Proof For the first assertion, by construction, $A^\diamondsuit \subseteq A_1$. For $k \geq 1$ the inclusion $A_{k+1} \subseteq A_k$ is clear. Assuming by induction $A^\diamondsuit \subseteq A_k$, we prove that $A^\diamondsuit \subseteq A_{k+1}$. Indeed, if $V_k = \emptyset$, we are done. If $V_k \neq \emptyset$ and $a \in A^\diamondsuit$, then $h_\alpha(a) \leq t_\alpha$ for each $\alpha \in V_k$. Hence $a \in A_{k+1}$, which shows that $A^\diamondsuit \subseteq A_{k+1}$. For the second assertion, when $\alpha \in W_k \setminus V_k$, by Lemma 2.2, one has $t_\alpha = 1$. Consequently, the inequality $h_\alpha(y) \leq t_\alpha$ is true for all $y \in A_k$ which yields (ii). The third assertion is obtained immediately from Lemma 2.2.
Assume now $V_k = \emptyset$ for some $k \geq 1$. Then $W_k \subseteq A^\diamondsuit$. By (i),

$$A^\diamondsuit \subseteq A_k = W_k^\diamondsuit \subseteq A^\diamondsuit,$$

and equality follows.

For the last assertion, let $A_0 := \cap_{k\geq 1} A_k$. Then, in view of (i), $\lim_{k\to\infty} A_k = A_0$. We show that $A_0 \cap \mathrm{int}\mathbb{R}^n_+ = A^\diamondsuit \cap \mathrm{int}\mathbb{R}^n_+$. Indeed, since $A^\diamondsuit \subseteq A_k$, we have $A^\diamondsuit \subseteq A_0$, which implies

$A^{\diamondsuit} \cap \operatorname{int}\mathbb{R}^n_+ \subseteq A_0 \cap \operatorname{int}\mathbb{R}^n_+$. For the converse inclusion, suppose to the contrary that there exists some $x \in A_0 \cap \operatorname{int}\mathbb{R}^n_+$ which does not belong to $A^{\diamondsuit}$. Since $x \in A_k$ and $\operatorname{Max}(A_k) = W_k$, there is some $\alpha^k \in W_k$ such that $x \leqq \alpha^k$. It is clear that $\alpha^k \in V_k$ because $x \notin A^{\diamondsuit}$. The sequence $\{\alpha^k\}_{k=1}^{\infty}$ being bounded, we may extract a subsequence $\left\{\alpha^{k(i)}\right\}_{i=1}^{\infty}$ that converges to some $\alpha \in A_0$. Then $x \leqq \alpha$ and $\alpha \in x + \mathbb{R}^n_+ \subseteq \operatorname{int}\mathbb{R}^n_+ + \mathbb{R}^n_+ \subseteq \operatorname{int}\mathbb{R}^n_+$. Moreover, $x \notin A^{\diamondsuit}$ implies $\alpha \notin A^{\diamondsuit}$ and hence $\alpha \in (A_0 \cap \operatorname{int}\mathbb{R}^n_+) \setminus A^{\diamondsuit}$. We have then

$$A^{\diamondsuit} \cap (\alpha + \mathbb{R}^n_+) = \emptyset.$$

As $A^{\diamondsuit}$ is compact, there is $\delta > 0$ such that

$$A^{\diamondsuit} \cap ((1-\delta)\alpha + \mathbb{R}^n_+) = \emptyset.$$

Let $i_0 \geq 1$ be such that

$$\left(1 - \frac{\delta}{4}\right)\alpha + \mathbb{R}^n_+ \subseteq \left(1 - \frac{\delta}{2}\right)\alpha^{k(i)} + \mathbb{R}^n_+ \subseteq (1-\delta)\alpha + \mathbb{R}^n_+ \quad \text{for } i \geq i_0.$$

Such i_0 exists because $\alpha^{k(i)} \to \alpha \in \operatorname{int}\left[\left(1 - \frac{\delta}{2}\right)\alpha + \mathbb{R}^n_+\right]$. Hence

$$A^{\diamondsuit} \cap \left[\left(1 - \frac{\delta}{2}\right)\alpha^{k(i)} + \mathbb{R}^n_+\right] = \emptyset \quad \text{for } i \geq i_0.$$

It follows from the definition of $A_{k(i)+1}$ that

$$A_{k(i)+1} \cap \left[\left(1 - \frac{\delta}{2}\right)\alpha^{k(i)} + \mathbb{R}^n_+\right] = \emptyset, \quad i \geq i_0.$$

Since the sequence $\{A_k\}_{k=1}^{\infty}$ is decreasing (by inclusion), $A_{k(i+1)} \subseteq A_{k(i)+1}$ and we also have

$$A_{k(i+1)} \cap \left[\left(1 - \frac{\delta}{2}\right)\alpha^{k(i)} + \mathbb{R}^n_+\right] = \emptyset, \quad i \geq i_0.$$

This is a contradiction as $\alpha^{k(i+1)}$ is a vertex of $A_{k(i+1)}$ and $\alpha^{k(i+1)} \in \left(1 - \frac{\delta}{4}\right)\alpha + \mathbb{R}^n_+ \subseteq \left(1 - \frac{\delta}{2}\right)\alpha^{k(i)} + \mathbb{R}^n_+$ for i sufficiently large. The proof is complete. □

We now apply the third assertion of Lemma 2.2 to present a practical way to compute the generating set W_{k+1} of A_{k+1}

Procedure (W):

Let $V_k = \{\alpha^1, \ldots, \alpha^p\}$ and set $W_k(0) = W_k$. Then a generating set of the set

$$A_k(1) := A_k \cap \left\{y \in \mathbb{R}^n_+ : h_{\alpha^1}(y) \leq t_{\alpha^1}\right\}$$

is given by

$$W_k(1) = \left\{V(\beta | t_{\alpha^1}\alpha^1) \colon \beta \in W_k(0)\right\}.$$

Continuing this process for $\alpha^2, \ldots, \alpha^p$, we obtain that a generating set of the set $A_{k+1} = A_k(p)$ is given by

$$W_k(p) = \{V(\beta | t_{\alpha^p}\alpha^p) \colon \beta \in W_k(p-1)\}.$$

Then A_{k+1} is generated by $W_k(p)$ and $W_{k+1} = \operatorname{Max}(W_k(p))$.

To understand the construction of the sets A_k we described above, let us consider the following example. The set A is given as in Fig. 1a. The first free disposal polyhedron A_1

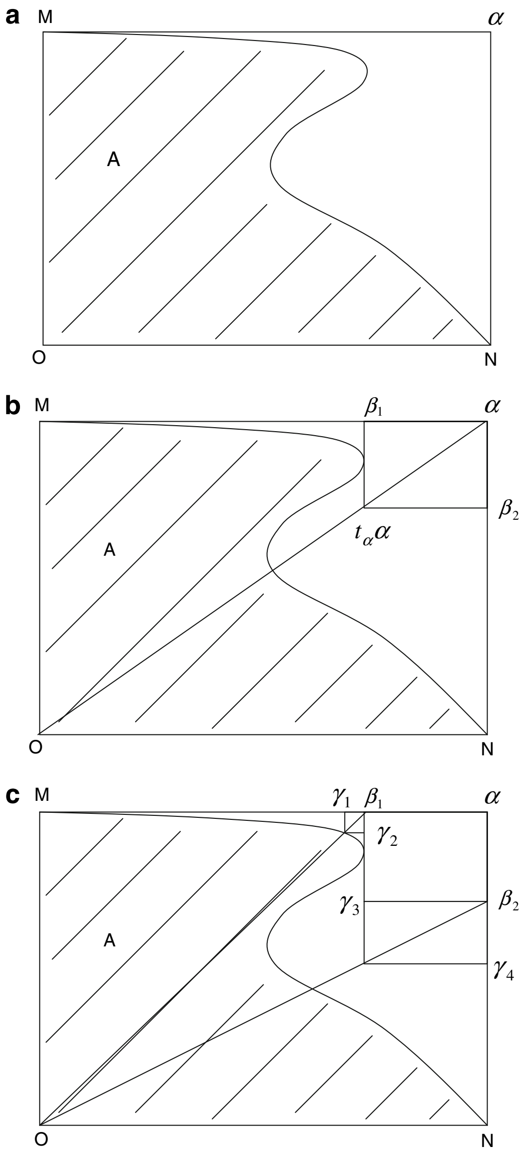


Fig. 1 (**a**) Construction of A_1, (**b**) construction of A_2, (**c**) construction of A_3

approximating A is the box $[MON\alpha]$, where α is found by solving (P_0). For the second step we solve (P_α), which gives us t_α, and obtain the free disposal nonconvex polyhedron A_2 generated by $W_2 = \{\beta_1, \beta_2\}$. Fig. 1b shows this construction. The next polyhedron A_3 is generated by $W_3 = \{\gamma_1, \gamma_2, \gamma_4\}$ (see Fig. 1c). Observe that the set $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ generates A_3 too, but γ_3 is not efficient, so it can be dropped from the consideration. It is also worthwhile noticing that $\lim_{k\to\infty} A_k \neq A^\diamondsuit$ in general.

3 Scalarizing functions

We shall denote by Λ the standard simplex of $\mathbb{R}^n$. It consists of all vectors $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_+$ with $\sum_{i=1}^n \lambda_i = 1$. The relative interior of Λ is denoted by $\mathrm{ri}(\Lambda)$.

For $A \in \mathcal{C}$ we define the function H_A on $\mathbb{R}^n_+$ by

$$H_A(y) := \sup_{\lambda \in \Lambda} \frac{h_\lambda(y)}{\max_{a \in A} h_\lambda(a)}.$$

Observe that for every $\lambda \in \Lambda$, the value $\max_{a \in A} h_\lambda(a)$ is strictly positive and actually one can find $\delta > 0$ such that $\max_{a \in A} h_\lambda(a) \geq \delta$ for all $\lambda \in \Lambda$, therefore the function H_A is well defined.

Lemma 3.1 *Let $A \in \mathcal{C}$. Then the function $\lambda \mapsto \max_{a \in A} h_\lambda(a)$ is continuous on Λ.*

Proof First we consider the case $\lambda \in \mathrm{ri}(\Lambda)$. It is clear that the function $h_\lambda(a)$ is continuous in both variables λ and a on $(\mathrm{int}\mathbb{R}^n_+) \times \mathbb{R}^n$. Moreover, as A is compact, the max-function $\max_{a \in A} h_\lambda(a)$ is continuous in $\lambda \in \mathrm{ri}(\Lambda)$. It remains to consider the case $\lambda = (\lambda_1, \ldots, \lambda_q, 0, \ldots, 0)$, with $\lambda_i > 0, i = 1, \ldots, q$ for some $q: 1 \leq q < n$. Let $\lambda^k \in \Lambda$ converge to λ. We wish to prove that

$$\limsup_{k \to \infty} \max_{a \in A} h_{\lambda^k}(a) \leq \max_{a \in A} h_\lambda(a) \leq \liminf_{k \to \infty} \max_{a \in A} h_{\lambda^k}(a). \tag{1}$$

Note that there is some $k_0 \geq 0$ such that $\lambda_i^k > 0$ for $i = 1, \ldots, q$ and $k \geq k_0$. We have then

$$\begin{aligned} h_\lambda(a) &= \min_{i=1,\ldots,q} \frac{a_i}{\lambda_i} \\ h_{\lambda^k}(a) &= \min_{i=1,\ldots,n} \left\{ \frac{a_i}{\lambda_i^k} : \lambda_i^k \neq 0 \right\} \leq \min_{i=1,\ldots,q} \frac{a_i}{\lambda_i^k}, \quad \text{for } k \geq k_0. \end{aligned}$$

Since A is compact, there is $\delta > 0$ such that $|a_i| \leq \delta$ for every $a = (a_1, \ldots, a_n) \in A$. For every $\epsilon > 0$ and $i = 1, \ldots, q$, when k is sufficiently large, we have

$$\frac{a_i}{\lambda_i^k} - \frac{a_i}{\lambda_i} \leq \epsilon a_i \leq \epsilon \delta.$$

It follows that

$$\begin{aligned} \limsup_{k \to \infty} \max_{a \in A} h_{\lambda^k}(a) &\leq \limsup_{k \to \infty} \max_{a \in A} \min_{i=1,\ldots,q} \left(\frac{a_i}{\lambda_i} + \frac{a_i}{\lambda_i^k} - \frac{a_i}{\lambda_i} \right) \\ &\leq \max_{a \in A} h_\lambda(a) + \epsilon \delta. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary small, we derive the first inequality of (1). For the second inequality of (1), observe that the function $h_\lambda(\cdot)$ being continuous on $\mathbb{R}^n_+$, there exists some $a^0 \in A$ such that

$$h_\lambda(a^0) = \min_{i=1,\ldots,q} \left(\frac{a_i^0}{\lambda_i} \right) = \max_{a \in A} h_\lambda(a). \tag{2}$$

For $\epsilon > 0$ sufficiently small, we can find $a' \in A\cap \operatorname{int}\mathbb{R}^n_+$ such that $\left|a'_i - a^0_i\right| \leq \epsilon$ for every $i = 1, \dots, n$. Then

$$h_{\lambda^k}(a') = \min\left\{\frac{a'_i}{\lambda^k_i}: i \in \{1, \dots, n\}, \lambda^k_i \neq 0\right\}.$$

For $i = q+1, \dots, n$, we have $\lambda^k_i \to 0$, while $a'_i > 0$. This implies that

$$h_{\lambda^k}(a') = \min_{i=1,\dots,q} \frac{a'_i}{\lambda^k_i}$$

for k sufficiently large. Consequently,

$$\begin{aligned}\liminf_{k\to\infty} \max_{a\in A} h_{\lambda^k}(a) &\geq \liminf_{k\to\infty} h_{\lambda^k}(a') \\ &\geq \liminf_{k\to\infty} \min_{i=1,\dots,q} \frac{a'_i}{\lambda^k_i} \\ &\geq h_\lambda(a^0) - \epsilon \max_{i=1,\dots,q} \frac{1}{\lambda_i}.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary small, we conclude

$$\liminf_{k\to\infty} \max_{a\in A} h_{\lambda^k}(a) \geq h_\lambda(a^0),$$

which together with (2) yields the second inequality of (1). The continuity of the function $\lambda \mapsto \max_{a\in A} h_\lambda(a)$ is proven. □

Notice that the conclusion of the above lemma is not true for any compact $A \subseteq \mathbb{R}^n_+$. Indeed, let A be a subset of $\mathbb{R}^2$ which consists of the simplex Λ and the point $(2, 0)$. For $\lambda^k = (1 - 1/k, 1/k)$ converging to $\lambda = (1, 0)$ we have

$$1 = \max_{a\in A} h_{\lambda^k}(a) < \max_{a\in A} h_\lambda(a) = 2$$

and so the function $\lambda \mapsto \max_{a\in A} h_\lambda(a)$ is not continuous on Λ.

Here is a simpler expression for the function H_A.

Lemma 3.2 *Let $A \in \mathcal{C}$, $y \in \mathbb{R}^n_+ \setminus \{0\}$ and $\lambda_y := \frac{y}{\sum_{i=1}^n y_i}$. Then*

$$H_A(y) = \frac{h_{\lambda_y}(y)}{\max_{a\in A} h_{\lambda_y}(a)}.$$

Proof Let us denote the function in the right hand side of the above equality by $\phi(y)$, which is also positively homogeneous. It follows from the definition of H_A that $\phi(y) \leq H_A(y)$. For the converse inequality, observe that $\phi(y) = \alpha$ implies that there is $a \in A$ such that $\alpha a \geqq y$. Then, it is obvious that $H_A(y) \leq \alpha$. □

Most useful properties of the function H_A are given in the next theorem.

Theorem 3.3 *Let $A, A_1, A_2 \in \mathcal{C}$. The following assertions hold:*

(i) *H_A is positively homogeneous, continuous and weakly monotonic on $\mathbb{R}^n_+$;*
(ii) *$H_A = H_{A^\diamond}$;*
(iii) *$H_{A_1}(y) \leq H_{A_2}(y)$ for every $y \in \mathbb{R}^n_+$ if and only if $A_2^\diamond \subseteq A_1^\diamond$;*
(iv) *For every $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda$, when $\epsilon > 0$ is sufficiently small one has*
$$\frac{1}{1+\frac{\epsilon}{\min\{\lambda_i : \lambda_i \neq 0\}\max\{h_\lambda(a) : a \in A\}}} H_A(\lambda) \leq H_{(A+\epsilon B_n)\cap\mathbb{R}^n_+}(\lambda) \leq H_A(\lambda);$$

(v) *Let* $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{C}$ *be a sequence of closed sets, H-converging to a closed set* $A \in \mathcal{C}$. *Then for every* $y \in \mathbb{R}^n_+$ *one has* $\lim_{k\to\infty} H_{A_k}(y) = H_A(y)$.

Proof For the first assertion we observe that h_λ is positively homogeneous, then so too is H_A. The continuity of H_A comes directly from Lemmas 3.1 and 3.2. The weak monotonicity of H_A is obtained from the same property of h_λ.
For the second assertion, we have evidently

$$\max_{a\in A} h_\lambda(y) \le \max_{a\in A^\diamondsuit} h_\lambda(y),$$

because $A \subseteq A^\diamondsuit$. Moreover, for every $a \in A^\diamondsuit$, there is $a' \in A$ such that $a' \geqq a$. Consequently, $h_\lambda(a') \ge h_\lambda(a)$ and $\max_{a\in A} h_\lambda(a) = \max_{a\in A^\diamondsuit} h_\lambda(a)$. By this equality (ii) follows. As to the third assertion, let $A_1, A_2 \in \mathcal{C}$ with $A_2^\diamondsuit \subseteq A_1^\diamondsuit$. In view of (ii) and Lemma 3.2 one has

$$H_{A_1}(y) = H_{A_1^\diamondsuit}(y) \le H_{A_2^\diamondsuit}(y) = H_{A_2}(y).$$

Conversely, assume that $H_{A_1}(y) \le H_{A_2}(y)$ for every $y \in I\!R^n_+$. In particular for $a \in A_2$, one has

$$H_{A_1}(a) \le H_{A_2}(a) \le 1. \tag{3}$$

If $a \notin A_1^\diamondsuit$, then $a \ne 0$ and $(a + I\!R^n_+) \cap A_1^\diamondsuit = \emptyset$. Then, one has

$$h_{\lambda_a}(z) < h_{\lambda_a}(a) \quad \text{for every} \quad z \in A_1^\diamondsuit.$$

Consequently,

$$H_{A_1}(a) > 1.$$

This contradicts (3), by which $A_2 \subseteq A_1^\diamondsuit$ and $A_2^\diamondsuit \subseteq A_1^\diamondsuit$ as well.

To prove the fourth assertion, let $a \in A$, $b \in B_n$ and $\epsilon > 0$ such that $a + \epsilon b \in I\!R^n_+$. Then one has

$$\begin{aligned} h_\lambda(a+\epsilon b) &= \min\left\{\frac{a_i+\epsilon b_i}{\lambda_i} : \lambda_i \ne 0\right\} \\ &\le \min\left\{\frac{a_i}{\lambda_i} : \lambda_i \ne 0\right\} + \frac{\epsilon}{\min\{\lambda_i : \lambda_i \ne 0\}} \\ &\le h_\lambda(a) + \frac{\epsilon}{\min\{\lambda_i : \lambda_i \ne 0\}}. \end{aligned}$$

This yields, in view of Lemma 3.2, that

$$\begin{aligned} H_{(A+\epsilon B_n)\cap I\!R^n_+}(\lambda) &= \frac{1}{\max_{z\in(A+\epsilon B_n)\cap I\!R^n_+} h_\lambda(z)} \\ &\ge \frac{1}{\max_{a\in A} h_\lambda(a) + \frac{\epsilon}{\min\{\lambda_i : \lambda_i \ne 0\}}} \\ &\ge \frac{1}{1 + \frac{\epsilon}{\min\{\lambda_i : \lambda_i \ne 0\}\max\{h_\lambda(a) : a\in A\}}} \frac{1}{\max_{a\in A} h_\lambda(a)} \\ &\ge \frac{1}{1 + \frac{\epsilon}{\min\{\lambda_i : \lambda_i \ne 0\}\max\{h_\lambda(a) : a\in A\}}} H_A(\lambda). \end{aligned}$$

The second inequality of (iv) follows from (iii) and the inclusion $A \subseteq (A + \epsilon B_n) \cap I\!R^n_+$. The last assertion is obtained from (iv). □

As we shall see later, the weak monotonicity of the function H_A allows us to obtain weakly efficient solutions of (VP) by minimizing the scalar composite function $H_A \circ f$ on X. For this reason one call it a scalarizing function associated to the set A. In the remaining part of this section, let $A \in \mathcal{C}$ and let A_k be the sequence of free disposal approximations of A described in Sect. 2. The scalarizing functions associated to A_k will recursively be computed. To this end, set

$$g_1(y) := \max\{h_{e_i}(y)/\alpha_i^0 : i = 1, \ldots, n\}.$$
$$g_k(y) := \max\{g_{k-1}(y), h_{t_\alpha\alpha}(y) : \alpha \in V_{k-1}\}.$$

for $k \geq 2$, and $y \in \mathbb{R}^n_+$.

Theorem 3.4 *The following assertions hold:*

(i) g_k *is continuous, positively homogeneous and weakly monotonic on* $\mathbb{R}^n_+$;
(ii) $g_k(y) = H_{A_k}(y)$ *for* $k \geq 1$ *and* $y \in \mathbb{R}^n_+$;
(iii) *For every* $y \in \mathbb{R}^n_+$, *the limit* $\lim_{k\to\infty} g_k(y)$ *exists and for* $y \in int\mathbb{R}^n_+$, $\lim_{k\to\infty} g_k(y) \leq 1$ *if and only if* $y \in A^\diamond$.

Proof The first assertion follows from the properties of the functions $h_\lambda(\cdot)$. For the second assertion, since g_k and H_{A_k} are positively homogeneous, it suffices to show that for $y \in \mathbb{R}^n_+$,

$$g_k(y) \leq 1 \quad \text{if and only if} \quad H_{A_k}(y) \leq 1. \tag{4}$$

We prove it by induction on k. For $k = 1$, we see that $g_1(y) \leq 1$ if and only if

$$y_i \leq \alpha_i^0, \quad i = 1, \ldots, n. \tag{5}$$

While the inequality $H_{A_1}(y) \leq 1$ is equivalent to the relation

$$h_\lambda(y) \leq \max_{a \in A_1} h_\lambda(a) \quad \text{for all} \quad \lambda \in \Lambda.$$

By choosing $\lambda = e_i$ in the latter relation we obtain (5) because $\max_{a\in A_1} h_{e_i}(a) = \alpha_i^0$. The converse is evident because if (5) is true, then $H_{A_1}(y) \leq 1$.

Assuming $g_{k-1}(y) = H_{A_{k-1}}(y)$ we now show that $g_k(y) = H_{A_k}(y)$ for every $y \in \mathbb{R}^n_+$. We first claim that

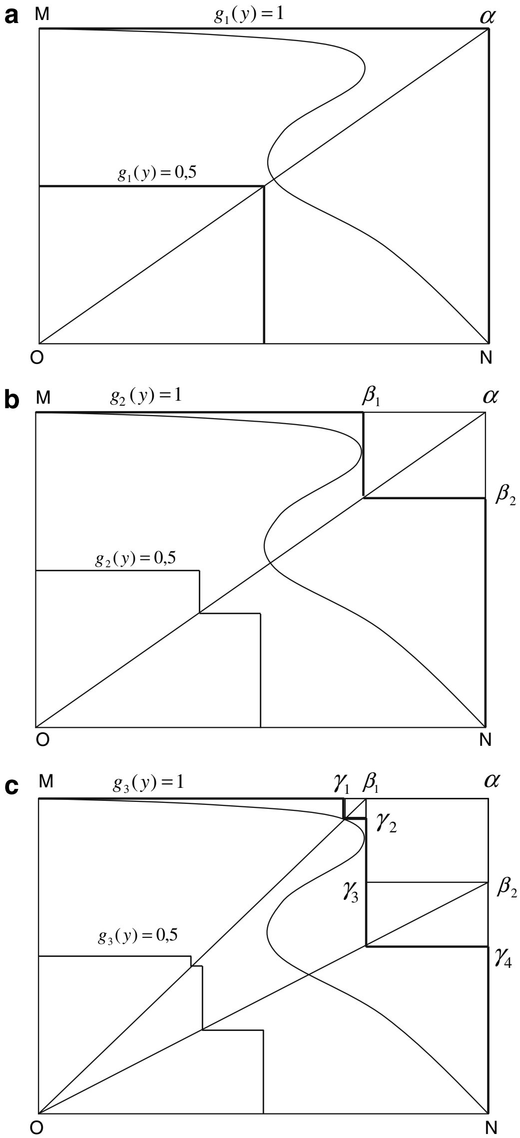
$$H_{A_k}(y) \leq 1 \quad \text{if and only if} \quad y \in A_k^\diamond. \tag{6}$$

Indeed, observe that for $z \in \mathbb{R}^n_+ \setminus \{0\}$, one has

$$(z + \mathbb{R}^n_+) \cap A_k^\diamond = \emptyset \quad \text{if and only if} \quad h_{\lambda_z}(z) > \max_{a \in A_k^\diamond} h_{\lambda_z}(a) = \max_{a \in A_k} h_{\lambda_z}(a),$$

where $\lambda_z = z/\sum_{i=1}^n z_i$. Hence (6) follows. To prove (4) we know by definition that $g_k(y) \leq 1$ if and only if $g_{k-1}(y) \leq 1$ and $h_{t_\alpha\alpha}(y) \leq 1$ for all $\alpha \in V_{k-1}$. The first inequality, by induction, is equivalent to $H_{A_{k-1}}(y) \leq 1$ which in its turn, by (6) is equivalent to $y \in A_{k-1}^\diamond$.

Fig. 2 (**a**) Level sets of g_1, (**b**) level sets of g_2, (**c**) level sets of g_3



The second relation can be rewritten as

$$h_\alpha(y) \le t_\alpha \quad \text{for all} \quad \alpha \in V_{k-1}.$$

By definition, these inequalities imply that $y \in A_k^\diamond$, and hence (4) holds.

The last assertion is obtained directly from Theorems 2.3(i), (v) and 3.3(iii) by observing that $g_k(y)$ is increasing and bounded. $\square$

With the data of the example given in Sect. 2, the level sets of the functions g_1, g_2 and g_3 are respectively illustrated in Fig. 2a–c.

4 Solving problem (VP)

In this section we wish to exploit the scalarizing functions g_k that we have constructed in Sect. 3 to solve problem (VP). Assume that $f(X)$ is a nonempty and compact set in the interior of $\mathbb{R}^n_+$ and throughout this section, we set $A = f(X)$. Consider the following scalarized problem, denoted by (P_k)

$$\begin{aligned} &\max \quad g_k \circ f(x) \\ &\text{subject} \quad \text{to } x \in X. \end{aligned}$$

The existence of optimal solutions of this problem as well as (VP) is guaranteed for instance when $f(X)$ is a compact set. We shall not return to this question, but concentrate our efforts to the links between optimal solutions of the scalarized problems and weakly efficient solutions of (VP) and their convergence. Recall that given a sequence of closed sets $\{D_k\}_{k=1}^{\infty}$, its upper limit in the sense of Kuratowski-Painleve is the set $\limsup_{k\to\infty} D_k$ of all possible limits of subsequences of $a_k \in D_k, k \geq 1$.

Here is the main result on the method we propose which leads to an algorithm to solve the problem (VP).

Theorem 4.1 *Assume that X is a nonempty and compact set, and that f is a continuous function with $f(X) \subseteq int\mathbb{R}^n_+$. Then the following assertions hold*:

(i) $S(g_k \circ f, X) = \{x \in X : g_k(f(x)) = 1\} \subseteq WS(f, X)$;
(ii) $\limsup_{k\to\infty} S(g_k \circ f, X) \subseteq WS(f, X)$;
(iii) $WMax(f(X)) \subseteq f\left[\limsup_{k\to\infty} S(g_k \circ f, X)\right] - \mathbb{R}^n_+$.

Proof For the first assertion we derive from Theorem 3.4 that

$$g_k(f(x)) = H_{A_k}(f(x)) \quad \text{for every} \quad x \in X.$$

Since $f(X) \subseteq A_k$, we have

$$h_\lambda(f(x)) \leq \max_{a\in A_k} h_\lambda(a) \quad \text{for each} \quad x \in X,$$

which shows that $g_k(f(x)) \leq 1$. By choosing $x_0 \in X$ that solves the problem

$$\max_{x\in X} g_1 \circ f(x),$$

we see that

$$1 \geq g_k(f(x_0)) \geq g_1(f(x_0)) = 1.$$

Hence the optimal value of problem (P_k) is equal to 1. Furthermore, let $x \in X$ with $g_k(f(x)) = 1$. There is some $\lambda \in \Lambda$ such that

$$h_\lambda(f(x)) = \max_{a\in A_k} h_\lambda(a).$$

Hence

$$h_\lambda(f(x)) = \max_{a\in f(X)} h_\lambda(a).$$

By the weak monotonicity of the function h_λ we conclude that x is a weakly efficient solution of (VP).

The second assertion is obtained from (i) and from the fact that $WS(f, X)$ is a closed set.

For the last assertion, let $y = f(x)$ be a weakly efficient point of $f(X)$. Since $f(X) \subseteq A_k$ and A_k is generated by the elements of W_k, there exists $\alpha^k \in W_k$ such that $y \leqq \alpha^k$ for every $k \geq 1$. Let $x^k \in X$ be a solution of (P_{α^k}). Then $f(x^k) \geqq t_{\alpha^k}\alpha^k$ and $x^k \in S(g_{k+1} \circ f, X)$. Here we have used the fact that $g_{k+1}(y) = \max\{g_k(y), h_{t_\alpha \alpha}(y) \colon \alpha \in W_k\}$ (see Theorem 3.4(ii)). By taking a subsequence if necessary, we may assume that $\alpha^k \to \alpha$, $x^k \to x^0$. Moreover, $y \in \operatorname{int}\mathbb{R}^n_+$ implies that $\alpha \in \operatorname{int}\mathbb{R}^n_+$. By this and Theorem 2.3 we assume that $\alpha \in A^\diamond$. Since $1 \geq t_{\alpha^k} = \max_{a \in f(X)} h_{\alpha^k}(a) \geq h_{\alpha^k}(\alpha)$, then $t_{\alpha^k} \to 1$. Hence,

$$y \leqq \lim_{k\to\infty} \alpha^k = \alpha = \lim_{k\to\infty} t_{\alpha^k}\alpha^k \leqq \lim_{k\to\infty} f(x^k) = f(x^0),$$

which completes the proof. □

We now are able to describe a general scheme of the algorithm for finding the weakly efficient solution set of problem (VP) which is based on the analysis above.

Step 1. (initialization) For $i = 1, \dots, n$ solve

$$\alpha_i = \max_{x\in X} f_i(x).$$

Find

$$S = \cup_{i=1}^n \{x \in X \colon f_i(x) = \alpha_i\}$$
$$E = \{f(x) \colon x \in S\}$$

Put $k = 1$, $W_0 = \emptyset$ and $W_1 = \{(\alpha_1, \dots, \alpha_n)\}$.

Step 2. For $\alpha \in W_k \setminus \cup_{i=0}^{k-1} W_i$, solve

$$t_\alpha = \max_{x\in X} h_\alpha(f(x)).$$

Compute

$$V_k = W_k \setminus \{\alpha \in \cup_{i=1}^k W_i \colon t_\alpha = 1\}.$$

Step 3. If $V_k = \emptyset$, stop. Otherwise

(3a) Find for $\alpha \in W_k \setminus \cup_{i=0}^{k-1} W_i$,

$$S(\alpha) = \{x \in X \colon h_{t_\alpha \alpha}(f(x)) = 1\}$$
$$E(\alpha) = \{f(x) \colon x \in S(\alpha)\}.$$

(3b) Set

$$S = S \cup \left\{ S(\alpha) \colon \alpha \in W_k \setminus \cup_{i=0}^{k-1} W_i \right\}$$
$$E = E \cup \left\{ E(\alpha) \colon \alpha \in W_k \setminus \cup_{i=0}^{k-1} W_i \right\}.$$

(3c) Determine W_{k+1} as described in Sect. 2, Procedure(W).

(3d) Put $k = k + 1$ and return to Step 2.

Let us point out two major properties of the algorithm.

(1) *Obtention of weakly efficient solutions and weakly efficient values.* At the kth iteration, the set S of Step 3 is exactly the solution set $S(g_{k+1} \circ f, X)$ and the set E is its set of values in the outcome space $\mathbb{R}^n$. Consequently,

$$S \subseteq WS(f, X) \quad \text{and} \quad E \subseteq WMax(f, X).$$

Indeed, by the definition of α_i, we have

$$g_1(f(x)) = \max\left\{\frac{f_i(x)}{\alpha_i}: i = 1, \ldots, n\right\}$$

and $g_1(f(x)) = 1$ if and only if $f(x) \in E$. For $k \geq 1$, one has that $g_{k+1}(f(x)) = 1$ which means that $x \in S(g_{k+1} \circ f, X)$, in view of Theorem 4.1, if and only if either $g_k(f(x)) = 1$, or $h_{t_\alpha \alpha}(f(x)) = 1$ for some $\alpha \in V_k$. Therefore, by induction, $g_{k+1}(f(x)) = 1$ if and only if $x \in S$ (of Step 3).

(2) *Convergence.* Denote the upper limit of the set E in Step 3 when k tends to ∞ by E_∞. Then

$$E_\infty \subseteq WMax(f, X) \subseteq E_\infty - \mathbb{R}^n_+.$$

This is the third assertion of Theorem 4.1. In particular, for every weakly efficient solution x of problem (VP) one can generate a sequence of weakly efficient solutions $\{x^k\}$ by the algorithm the limit of which dominates x, i.e. $f(\lim_{k\to\infty} x^k) \geqq f(x)$.

The following comments are useful in numerical implementation of the algorithm.

(a) *Collecting the optimal solutions and optimal values.* In general the maximization problems occurring in the algorithm are neither linear, nor convex, therefore most existing solvers offer, for each $\alpha \in V_k$, one solution x^α and its value $f(x^\alpha)$ only. Consequently, the following modifications are to be taken into account when coding the program.

- In Step 1 the set S consists of n solutions $x^1, \ldots, x^n$ with $f(x^i) = \alpha_i$ which are obtained by solving the problem of maximizing f_i over X.
- In Step 3 the set $S(\alpha)$ consists of one solution x^α with $h_{t_\alpha \alpha}(f(x^\alpha)) = 1$ and the set $E(\alpha) = \{f(x^\alpha)\}$.

We notice also that in practice it is quite often that the solution set $S(\alpha)$ in Step 3 is a singleton, or is not a singleton, but the value set $E(\alpha)$ is (most of examples given in the existing literature on the topic have this property). Here are some particular cases we cite without going into details.

(i) *Strictly quasiconcave problems.* The problem (VP) is strictly quasiconcave if X is a convex set and f is strictly quasiconcave, that is, $f_i(tx + (1-t)y) > \min\{f_i(x), f_i(y)\}$ when $x, y \in X, x \neq y$ and $0 < t < 1$, $i = 1, \ldots, n$. When (VP) is strictly quasiconcave, the set $S(\alpha)$ is a singleton, and hence so is the set $E(\alpha)$.

(ii) *Strictly quasiconcave-like problems.* The problem (VP) is strictly quasiconcave-like if $x, y \in X$ with $f(x) \neq f(y)$ there exists some $z \in X$ such that $f_i(z) > \min\{f_i(x), f_i(y)\}$, $i = 1, \ldots, n$. When (VP) is strictly quasiconcave-like, the set $S(\alpha)$ is not necessarily a singleton, but the set $E(\alpha)$ is. The convergence property (2) remains true, which means that all weakly efficient values can numerically be obtained. However, not all weakly solutions can be generated because at Step 3, for each α, only one solution x^α is stocked in the set $S(\alpha)$. Notice also that strictly quasiconcave problems are strictly quasiconcave-like, but the converse is not true, and for these problems, the weakly efficient solutions are efficient.

Note that the solution set S obtained in Step 3 forms a portion of weakly efficient solutions which is the best in the following sense. For $\alpha \in W_k$, define a new norm on $\mathbb{R}^n$ by

$$\|y\| = \max\left\{\frac{|y_i|}{\alpha_i}: i = 1, \ldots, n\right\}.$$

Then the value $f(x^{\alpha})$ is a nearest point of the set $f(X)$ to the reference point α with respect to this norm. In the terminology of multicriteria decision making [25], the collection of these solutions x^{α} represents the best compromise solution set of the weakly efficient solutions of the problem (VP) with respect to the targets formed by the generating set W_k of the free disposal outer approximation A_k of $f(X)$.

(b) *Stopping criterion.* Without a particular structure of the data f and X, the stopping criterion of Step 3 hardly holds. In such situations one may choose a priori a small positive number ϵ and set $V_k(\epsilon) = W_k \setminus \left\{\alpha \in \cup_{i=1}^{k} W_k : t_\alpha \geq 1 - \epsilon\right\}$. Then one stops the algorithm as soon as $V_k(\epsilon)$ is empty. We claim that for $\epsilon > 0$, the algorithm terminates after a finite number of iterations. Indeed, notice first that since $f(X) \subseteq \mathrm{int}\mathbb{R}^n_+$ one finds a positive number δ such that $f(X) \subseteq (\delta, \ldots, \delta) + \mathbb{R}^n_+$. Now suppose to the contrary that there exists $\alpha^k \in W_k$ such that $t_{\alpha^k} < 1 - \epsilon$ for every $k \geq 1$. Without loss of generality one may assume that α^k and t_{α^k} converge respectively to $\overline{\alpha} \in [f(X)]^{\diamondsuit}$ and $\overline{t} \leq 1 - \epsilon$ as k tends to ∞. We have then on one hand

$$f(X) \cap \left(t_{\alpha^k}\alpha^k + int\mathbb{R}^n_+\right) = \emptyset \quad \text{for all} \quad k \geq 1,$$

which implies

$$f(X) \cap ((1 - \epsilon/2)\overline{\alpha} + \mathbb{R}^n_+) = \emptyset. \tag{7}$$

On the other hand, by the construction of W_k, every element $\alpha \in W_k$ verifies the inequality $\alpha \geqq (\delta, \ldots, \delta)$. Therefore, $\overline{\alpha} \geqq (\delta, \ldots, \delta)$, and in particular, $\overline{\alpha} \in [f(X)]^{\diamondsuit} \cap \mathrm{int}\mathbb{R}^n_+$. This and (7) contradict the conclusion of Theorem 2.3(v).

(c) *Explicit form of the key program in the algorithm.* The problem, noted (P_α), that one has to repeatedly solve in Step 3 is the following:

$$\max_{x \in X} h_\alpha(f(x)),$$

where α is a strictly positive vector. It can be written in an explicit form as follows

$$\max_{x \in X} \min \left\{\frac{f_i(x)}{\alpha_i} : i = 1, \ldots, n\right\}.$$

If it happens that f_i are concave functions and X is a convex set, then we deal with a concave maximization problem and convex optimization techniques can be applied to solve it.

(d) *Bi-criteria problems.* For $n = 2$, the procedure to compute W_{k+1} is very simple. To obtain it suffices to compute the sets $V(\alpha \,|t_\alpha\alpha)$ for $\alpha \in W_k$ because the inequality $t_\alpha\alpha < \beta$ is impossible when $\beta \neq \alpha$ so that $V(\beta \,|t_\alpha\alpha) = \{\beta\}$ for $\beta \in W_k \setminus \{\alpha\}$.

5 Numerical examples

To perform a few preliminary computational examples we have used Matlab Optimization Toolbox. By our experience, the results we obtained by the help of the Optimization Toolbox are not fully satisfactory when the number of variables m is large and when the objective functions are of bad behavior, which is a common feature of nonconvex optimization. In order to improve the accuracy, at each optimization process several initial points were generated and only the best solutions were kept. At the third step, we compared the value $f(x^{\alpha})$ of the current solution x^{α} with all values of E previously computed, and so we could avoid error accumulation. Another problem is that the computing time increases rapidly with the number of objective functions: at the kth iteration, one may have to solve up to n^{k-1} minimax problems. Here n^{k-1} is the maximum number of the vertices of the set $W_k \setminus \cup_{i=0}^{k-1} W_i$. To decrease the computing time and better control the distribution of approximation points of the efficient set, we use also $V_k(\epsilon) = W_k \setminus \{\alpha \in \cup_{i=1}^{k} W_k : t_\alpha \geq 1-\epsilon\}$ instead of V_k when computing the set W_{k+1} by procedure (W) (Step 3(c)). Namely, those $\alpha \in W_k$ with $t_\alpha \geq 1-\epsilon$ will be dropped from the set of vertices that generates the new W_{k+1}.

5.1 Biobjective problems

5.1.1 Example 1

Consider the following biobjective problem:

$$\max\left(3-\sqrt{x_1}, 3-\sqrt{x_2}\right).$$
$$s.t.\ (x_1, x_2) \in [0, 8.99]^2, \quad x_1 + x_2 \geq 5.$$

In this example the biobjective function is not concave and the constraints are linear. With $\epsilon = 0.02$, the algorithm stops after seven iterations. The subsets of E obtained during the process are illustrated in Fig. 3.

5.1.2 Example 2

Consider the following biobjective problem:

$$\max\,(x_1, x_2)$$

$$s.t.\ (x_1, x_2) \in [0.01, 1]^2, (x_2 - 0.5x_1)(4x_1 - x_2) \leq 0.$$

This problem is of particular structure because $[f(X)]^\diamondsuit$ is a finitely generated free disposal set. Therefore, the stopping criterion of Step 3 is verified after a finite number of iterations and we can set $\epsilon = 0$. Indeed, for $k = 1$, we have $\alpha = (1, 1)$, $A_1 = [0, 1]^2$. By solving problem (P_α), we obtain $t_\alpha = 0.5$, $V_1 = \{\alpha\}$, $A_2 = A_1 \setminus [0.5, 1]^2$ and $W_2 = \{\beta_1, \beta_2\} = \{(0.5, 1), (1, 0.5)\}$. At the next step, we have $t_{\beta_1} = 0.5$, $t_{\beta_2} = 1$, $V_2 = \{\beta_1\}$, $A_3 = A_1 \setminus [0.25, 1] \times [0.5, 1]$ and $W_3 = \{\gamma_1, \gamma_2, \gamma_3\} = \{(0.25, 1), (0.5, 0.5), (1, 0.5)\}$. Finally, we get $t_{\gamma_1} = t_{\gamma_2} = 1$, which means $V_3 = \emptyset$ and $A_3 = A^\diamondsuit$. Note that $\gamma_3 = \beta_2$ which means that t_{γ_3} has already been computed at the previous step. Thus, the algorithm terminates after three iterations (see Fig. 4a).

5.1.3 Example 3

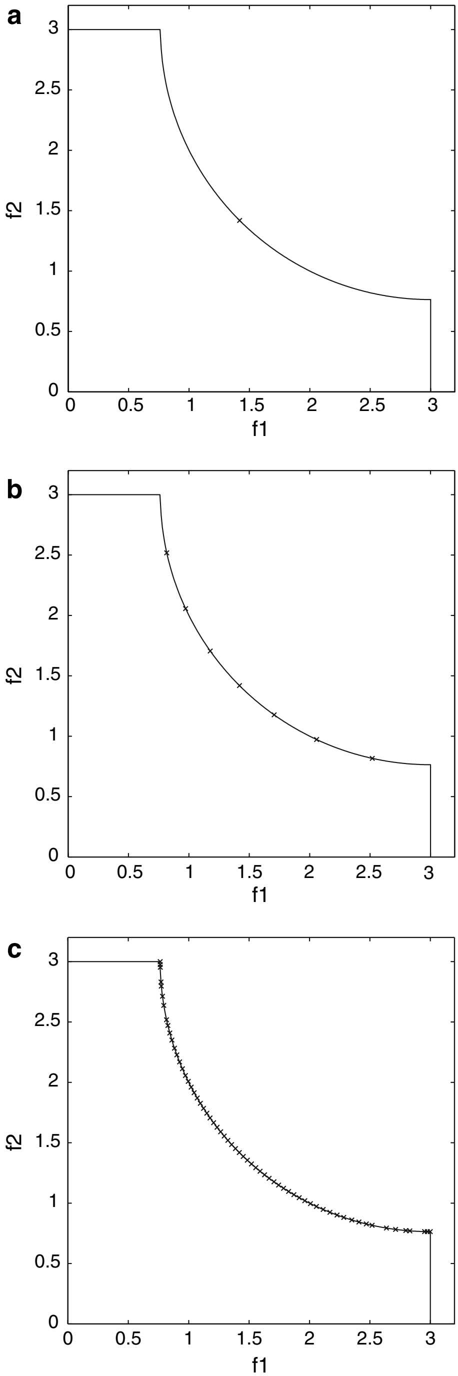
Consider the following problem:

$$\max\,(x_1, x_2)$$

$$s.t.\ (x_1, x_2) \in [0.01, +\infty[^2, x_1^2 + x_2^2 - 25 \leq 0, 1 - (x_1 - 4)^2 - (x_2 - 2)^2 \leq 0.$$

This problem is of bad structure because the constraint set is not convex and the solution set is not connected. With $\epsilon = 0.05$ after ten iterations, we obtain the subset of E illustrated in Fig. 4b. Notice that $\min\{t_\alpha : \alpha \in W_{10}\} \approx 1 - \epsilon = 0.95$, while the average of these t_α is about 0.9986. This means that a large majority of α at the last iteration provides elements of the solution set which are very closed to those computed before.

Fig. 3 $f(X)$ is presented by the solid line; x-marks describe the elements of the subset of E computed by the algorithm after (**a**) 1 iteration, (**b**) 3 iterations, (**c**) 7 iterations



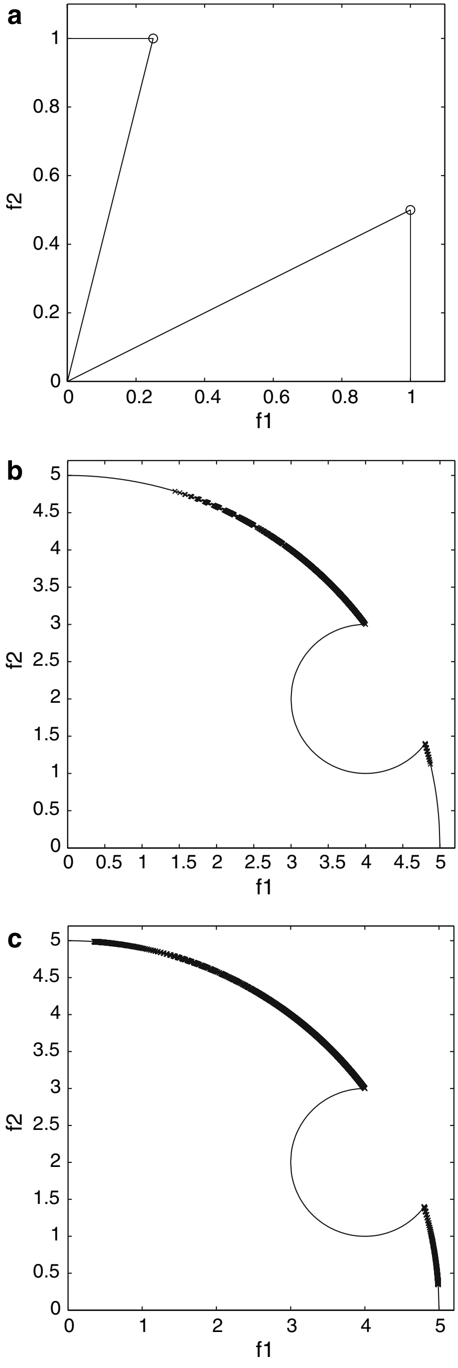


Fig. 4 (**a**) Example 2. $f(X)$ is presented by the solid line; o-marks describe the elements of the subset of E computed by the algorithm. (**b**) Example 3, 10 iterations. x-marks describe the elements of the subset of E computed by the algorithm. (**c**) Example 3, 250 iterations with the use of $V_k(\epsilon)$ in computing W_{k+1}

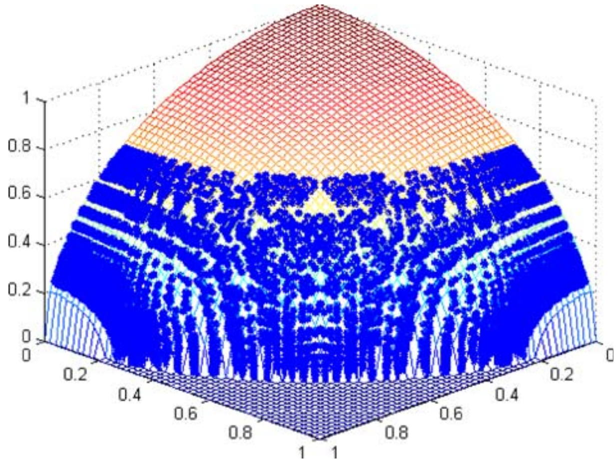


Fig. 5 Example 5. The mesh grid is the unit sphere in $\mathbb{R}^3_+$ and the x-marks are elements of the subset of E computed by the algorithm after 43 iterations

By using $V_k(\epsilon)$ to compute W_{k+1} as explained in the beginning of Sect. 5, with $\epsilon = 0.002$, the algorithm stops after 250 iterations and we obtain the subset of E illustrated in Fig. 4c. The efficiency of this use of $V_k(\epsilon)$ in computing W_{k+1} is shown by the fact that more computing time has been necessary to generate Fig. 4b than Fig. 4c where the approximation of the efficient set is better.

5.2 Three-objective problems

5.2.1 Example 4

Consider the following problem:

$$\max\,(x_1, x_2, x_3)$$

$$s.t.\ (x_1, x_2, x_3) \in [0.01, 1]^3\,, (x_2 - 0.5x_1)(4x_1 - x_2) \le 0.$$

This problem with three objective functions is very similar to the one of Example 2. It is worthwhile noticing that the stopping criterion (with $\epsilon > 0$) would never hold if the constraint $(x_1, x_2, x_3) \in [0, 1]^3$ were imposed instead of $(x_1, x_2, x_3) \in [0.01, 1]^3$. With $\epsilon = 0.01$, the algorithm terminates after nine iterations. Notice that the solutions we obtained in this example are weakly efficient only.

5.2.2 Example 5

Consider the following problem:

$$\max\,(x_1, x_2, x_3)$$

$$s.t.\ (x_1, x_2, x_3) \in [0.01, +\infty[^3, x_1^2 + x_2^2 + x_3^2 - 1 \le 0, x_3^2 - x_1^2 - x_2^2 \le 0.$$

By using $V_k(\epsilon)$ to compute W_{k+1}, with $\epsilon = 0.03$, the algorithm terminates after 43 iterations and we obtain the subset of E illustrated in Fig. 5.

6 Conclusion

The method presented in this paper is aimed at solving nonconvex multiobjective problems. It is based on a particular outer approximation of the outcome set $f(X)$ by free disposal polyhedra. The convergence result (Theorem 4.1) presents the main advantage of our approach over the existing methods we are aware of. The preliminary work on computational experiments proves the practicability of the method for small size problems. As we have noticed, the Optimization Toolbox, which is at our disposal, is much less efficient for nonconvex models with a big number of variables. We believe that global optimization solvers which are able to solve more efficiently scalar nonconvex models with a bigger number of variables, could allow us to treat multiobjective problems of larger size. This of course needs further investigation.

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